Præcipuarum Functionum Trigonometricarum per Analysin Infinitorum Explicatio.

Dissertatio,

Quam,

Consentiente Ampl. Facultate Philosophica

Præside

JONA BRAG,

Astron. Prof. Reg. et Ord.,

pro Gradu obtinendo

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I.

Functiones Goniometricæ.

Probl. 1. *Datis duobus arcubus, alterutriusque sinu, invenire sinum summæ et sinum differentiæ arcuum.*

$\widehat{\alpha}$ sit $u = fx = \sin x$; $u^1 = f(x+y) = \sin(x+y)$, habebimus mox:

$\frac{du}{dx} = \cos x$; $\frac{d^2u}{dx^2} = -\sin x$; $\frac{d^3u}{dx^3} = -\cos x$; $\frac{d^4u}{dx^4} = \sin x$; $\frac{d^5u}{dx^5} = \cos x$, etc.; quæ in formula Tayloriana rite disposita dabunt:

$$u^1 = \sin(x+y) = \sin x + y\cos x - \frac{y^2 \sin x}{1.2.} - \frac{y^3 \cos x}{1.2.3.} + \frac{y^4 \sin x}{1.2.3.4.} + \frac{y^5 \cos x}{1.2.3.4.5.} - \ldots.$$

$\widehat{\beta}$ Sit iterum $u = fx = \sin x$, u^1 vero $= f(x-y)$, erit ex $\widehat{\alpha}$, y negativo facto, $u^1 = \sin(x-y) = \sin x - y\cos x - \frac{y^2 \sin x}{1.2} + \frac{y^3 \cos x}{1.2.3} + \frac{y^4 \sin x}{1.2.3.4}$

$$-\frac{y^5 \cos x}{1.2.3.4.5} - + + \ldots$$

Probl. 2. *Datis duobus arcubus, alterutriusque cosinu, invenire cosinum summæ et cosinum differentiæ arcuum.*

Sumtis exponentibus differentialibus, ope ejusdem theorematis, mox obtinebitur:

$$\overset{\frown}{\alpha}\ u^1 = cos(x+y) = cos x - y\,sin\,x - \frac{y^2 cos\,x}{1.2} - + \frac{y^3 sin\,x}{1.2.3} + \frac{y^4 cos\,x}{1.2.3.4} - + + \ldots\ldots;$$

atque y in negativum mutato,

$$\overset{\frown}{\beta}\ u^1 = cos(x-y) = cos\,x + y sin x - \frac{y^2 cos x}{1.2} - \frac{y^3 sin x}{1.2.3} + \frac{y^4 cos\,x}{1.2.3.4} + - - \ldots.$$

Probl. 3. *Datis duobus arcubus, alterutriusque tangente, invenire tangentem summæ et tangentem differentiæ arcuum.*

$\overset{\frown}{\alpha}$ Sit $u = f\,x = tang\,x;\ u^1 = f(x+y) = tang(x+y);$ erit

$$\frac{du}{dx} = 1 + u^2;\ \frac{d^2u}{dx} = 2udu = 2u(1+u^2)\,dx,\ sive$$

$$\frac{d^2u}{dx^2} = 2u + 2u^3;\ \frac{d^3u}{dx^3} = (2+6u^2)(1+u^2) = 2 + 8u^2 + 6u^4;$$

$$\frac{d^4u}{dx^4} = (16u + 24u^3)(1+u^2) = 16u + 40u^3 + 24u^5;$$

$$\frac{d^5u}{dx^5} = 16 + 120u^2 + 120u^4)(1+u^2) = 16 + 136u^2 + 240u^4 + 120u^6;$$

$$\frac{d^6u}{dx^6} = (272u + 960u^3 + 720u^5)(1+u^2) = 272u + 1232u^3 + 1680u^5 + 720u^7;$$

$$\frac{d^7u}{dx^7}(272 + 3696u^2 + 8400u^4 + 5040u^6)(1+u^2) = 272 + 3968u^2 + 12096u^4 +$$

$13440u^6 + 5040u^8$; quibus, formulæ Tayl. adplicatis, habebitur:

$$u^1 = tang(x+y) = u + (1+u^2)y + (u+u^3)y^2 + \frac{(1+4u^2+3u^4)}{1.3}y^3$$

$$+ \frac{(2u+5u^3+3u^5)}{1.3}y^4 + \frac{(2+17u^2+30u^4+15u^6)}{1.3.5}y^5$$

$$+ \frac{(17u+77u^3+105u^5+45u^7)y^6}{1.3.5.3} + \frac{17+248u^2+756u^4+840u^6+315u^8)y^7}{1.3.5.7.3.} + \ldots$$

$\overset{\frown}{\beta}$ Si y negative sumatur, debita mutatione signorum facta, sine ullo negotio ex $\overset{\frown}{\alpha}$ obtinebitur $tang(x-y)$.

Probl. 4. *Dato arcu invenire Sinum.*

Si in inventa functione $sin\,(x+y)$, Probl. 1, *primum* arcum x cifræ æquaveris, *deinde* x, pro residuo y, scripseris; sive, quod idem est, obtentis exponentibus differentialibus, ponendo primum $x=o$, deinde scribendo x pro y, Taylorianum theorema Mac-Lauriniano restrinxeris, habebis:

$$sin\,x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7}+\ldots.$$

Probl. 5. *Dato arcu invenire cosinum.*

Scilicet iisdem conditionibus, quas modo nominavimus, e probl. 2 prodibit:

$$cos\,x=1-\frac{x^2}{1.2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6}+\frac{x^8}{1.2.3.4.5.6.7.8}-\ldots.$$

Probl. 6. *Dato arcu invenire tangentem.*

Emerget nempe, eadem via, e problemate tertio:

$$tang x=x+\frac{x^3}{3}+\frac{2x^5}{1.3.5}+\frac{17x^7}{1.3.5.3.7}+\frac{62x^9}{1.3.5.3.7.9}+\ldots.$$

Scholion. Ut commodior, ita frequentior, problematum 4:ti et 5:ti, quam 6:ti usus; ideoque, etsi series, lege sua, magis inde innotesceret, minor ejus ulterius hic evolvendæ necessitas.

Probl. 7. *Dato arcu invenire cotangentem.*

$$cot\,x=\frac{1}{x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\frac{62x^9}{2835}+\ldots}$$

$$=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{45}-\frac{2x^5}{945}-\frac{x^7}{4725}-\ldots$$

$$=\frac{1}{x}-\frac{x}{3}-\frac{x^3}{1.3.5.3}-\frac{2x^5}{1.3.5.3.7.3}-\frac{x^7}{1.3.5.3.7.3.5}-\ldots$$

Probl. 8. *Dato arcu invenire sinum versum.*

$$sin\ vers\ x=1-cos\,x=1-\left(1-\frac{x^2}{2}+\frac{x^4}{1.2.3.4}-\frac{x^6}{1.2.3.4.5.6.}+\ldots\right);$$

$$(probl.\ 5);=\frac{x^2}{2}-\frac{x^4}{1.2.3.4}+\frac{x^6}{1.2.3.4.5.6}-\ldots.$$

Probl. 9. *Dato arcu invenire secantem.*

$$sec\, x = \frac{1}{1 - \frac{x^2}{2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \frac{x^8}{1.2.3.4.5.6.7.8} - \dots}$$

$$\text{(probl. 5); } = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{1735x^8}{40320} + \dots$$

$$= 1 + \frac{x^2}{2} + \frac{5x^4}{1.2.3.4} + \frac{61x^6}{1.2.3.4.5.6} + \frac{1735x^8}{1.2.3.4.5.6.7.8} + \dots$$

Potuisset, sine ope problematis quinti, in morem priorum ita solvi:

$$u = sec\, x;\ u^1 = sec(x+y)$$

$$\frac{du}{dx} = \frac{sin x}{cos^2 x};\ \frac{d^2u}{dx^2} = \frac{cos^2 x + 2\, sin^2 x}{cos^3 x} = \frac{1 + sin^2 x}{cos^3 x};$$

$$\frac{d^3u}{dx^3} = \frac{2 cos^2 x\, sin x + 3 sin x + 3 sin^3 x}{cos^4 x} = \frac{(2 + sin^2 x) sin\, x + 3 sin\, x}{cos^4 x} = \frac{5 sin\, x + sin^3 x}{cos^4 x};$$

$$\frac{d^4u}{dx^4} = \frac{(5 + 3 sin^2 x) cos^2 x + 4 sin^4 x + 20 sin^2 x}{cos^5 x}$$

$$= \frac{5 + 3 sin^2 x - 5 sin^2 x - 3 sin^4 x + 4 sin^4 x + 20 sin^2 x}{cos^5 x} = \frac{5 + 18 sin^2 x + sin^4 x}{cos^5 x};$$

$$\frac{d^5 u}{d x^5} = \frac{(36 sin x + 4 sin^3 x) cos^2 x + 25 sin x + 90 sin^3 x + 5 sin^5 x}{cos^6 x}$$

$$= \frac{36 sin x + 4 sin^3 x - 36 sin^3 x - 4 sin^5 x + 25 sin x + 90 sin^3 x + 5 sin^5 x}{cos^6 x}$$

$$= \frac{61\, sin x + 58\, sin^3 x + sin^5 x}{cos^6 x};$$

$$\frac{d^6u}{dx^6} = \frac{(61 + 174 sin^2 x + 5 sin^4 x) cos^2 x + 366 sin^2 x + 348 sin^4 x + 6 sin^6 x}{cos^7 x}$$

$$= \frac{61 + 174 sin^2 x + 5 sin^4 x - 61 sin^2 x - 174 sin^4 x - 5 sin^6 x + 366 sin^2 x + 348 sin^4 x \quad sin^6 x}{cos^7 x}$$

$$= \frac{61 + 479 sin^2 x + 179 sin^4 x + sin^6 x}{cos^7 x};$$

Posito igitur $x=o$, et deinde adsumto pro y, habebimus

$$secx=1+\frac{x^2}{2}+\frac{5x^4}{1.2.3.4}+\frac{61x^6}{1.2.3.4.5.6}+\dots$$

Schol. Cosecantem utroque modo inveniri manifestum.

Prob. 10. *Dato sinu invenire arcum.*

$u=\text{arc}\ sinx$; $u^1=\text{arc}\ sin(x+y)$; ....

$$\frac{du}{dx}=\frac{1}{(1-x^2)^{\frac{1}{2}}};\ \frac{d^2u}{dx^2}=\frac{x}{(1-x^2)^{\frac{3}{2}}};$$

$$\frac{d^3u}{dx^3}=\frac{(1-x^2)^{\frac{3}{2}}+3x^2(1-x^2)^{\frac{1}{2}}}{(1-x^2)^3}=\frac{1+2x^2}{(1-x^2)^{\frac{5}{2}}};$$

$$\frac{d^4u}{dx^4}=\frac{4x(1-x^2)^{\frac{5}{2}}+5x(1+2x^2)(1-x^2)^{\frac{3}{2}}}{(1-x^2)^5}$$

$$=\frac{4x-4x^3+5x+10x^3}{(1-x^2)^{\frac{7}{2}}}=\frac{9x+6x^3}{(1-x^2)^{\frac{7}{2}}};$$

$$\frac{d^5u}{dx^5}=\frac{(9+18x^2)(1-x^2)^{\frac{7}{2}}+(63x^2+42x^4)(1-x^2)^{\frac{5}{2}}}{(1-x^2)^7}$$

$$=\frac{9+18x^2-9x^2-18x^4+63x^2+42x^4}{(1-x^2)^{\frac{9}{2}}}=\frac{9+72x^2+24x^4}{(1-x^2)^{\frac{9}{2}}};$$

$$\frac{d^6u}{dx^6}=\frac{(144x+96x^3)(1-x^2)^{\frac{9}{2}}}{(1-x^2)^9}+\frac{9x(9+72x^2+24x^4)(1-x^2)^{\frac{7}{2}}}{(1-x^2)^9}$$

$$=\frac{144x+96x^3-144x^3-96x^5+81x+648x^3+216x^5}{(1-{}^2)^{\frac{11}{2}}}=\frac{225x+600x^3+120x^5}{(1-x^2)^{\frac{11}{2}}};$$

$$\frac{d^7u}{dx^7}=\frac{(225+1800x^2+600x^4)(1-x^2)^{\frac{11}{2}}}{(1-x^2)^{11}}+\frac{11x(225+600x^3+120x^5)(1-x^2)^{\frac{9}{2}}}{(1-x^2)^{11}}$$

$$=\frac{225+1800x^2+600x^4-225x^2-1800x^4-600x^6}{(1-x^2)^{\frac{13}{2}}}+\frac{2475x^2+6600x^4+1320x^6}{(1-x^2)^{\frac{13}{2}}}$$

$$=\frac{225+4050x^2+5400x^4+720x^6}{(1-x^2)^{\frac{13}{2}}}.$$

Limitatione igitur et substitutione solita facta emerget:

$$u=x+\frac{x^3}{1.2.3}+\frac{3x^5}{2.4.5}+\frac{3.5x^7}{2.4.6.7}+\frac{3.5.7x^9}{2.4.6.8.9}+\ldots$$

vel, si arcus ipse vocetur x,

$$x=\sin x+\frac{\sin^3 x}{1.2.3}+\frac{3\sin^5 x}{2.4.5}+\frac{3.5\sin^7 x}{2.4.6.7}+\frac{3.5.7\sin^9 x}{2.4.6.8.9}+\ldots,$$

lege indubia.

Schol. Nisi, qua coepimus, pergere voluissemus via, ut, vel ad arc $\sin(x+y)$, secundum theorema Taylorianum determinandum, si placeret, omnia parata essent, commodius quidem ad hoc solvendum problema per *Methodum Reversionis Serierum* fuisset iter.

Dederat resolutio problematis quarti:

$$\sin x=x-\frac{x^3}{1.2.3}+\frac{x^5}{1.2.3.4.5}-\frac{x^7}{1.2.3.4.5.6.7}+\ldots;$$

quamobrem, si ponatur $x=\sin x+\alpha\sin^3 x+\beta\sin^5 x+\gamma\sin^7 x+\ldots,$

$$\begin{aligned}
\text{erit } -\frac{x^3}{1.2.3} &= -\frac{\sin^3 x}{1.2.3}-\frac{3\alpha\sin^5 x}{1.2.3.}-\frac{3\alpha^2\sin^7 x}{1.2.3} \\
&\qquad\qquad\qquad\qquad -\frac{3\beta\sin^7 x}{1.2.3} \\
+\frac{x^5}{1.2.3.4.5} &= \qquad\quad +\frac{\sin^5 x}{1.2.3.4.5}+\frac{5\alpha\sin^7 x}{1.2.3.4.5} \\
-\frac{x^7}{1.2.3.4.5.6.7} &= \qquad\qquad\qquad -\frac{\sin^7 x}{1.2.3.4.5.6.7};
\end{aligned}$$

sive

$$\left.\begin{matrix}\alpha \\ -\frac{1}{1.2.3}\end{matrix}\right\}\sin^3 x \quad \left.\begin{matrix}+\beta \\ -\frac{3\alpha}{1.2.3} \\ +\frac{1}{1.2.3.4.5}\end{matrix}\right\}\sin^5 x \quad \left.\begin{matrix}+\gamma \\ -\frac{3\alpha^2}{1.2.3} \\ -\frac{3\beta}{1.2.3} \\ +\frac{5\alpha}{1.2.3.4.5} \\ -\frac{1}{1.2.3.4.5.6.7}\end{matrix}\right\}\sin^7 x=0$$

Hinc

$$\alpha=\frac{1}{1.2.3};\ \beta=\frac{3}{2.4.5};\ \gamma=\frac{3.5}{2.4.6.7};\ \text{et sic in ceteris.}$$

Prodibit igitur, ut antea,

$$x=sin\,x+\frac{sin^3x}{1.2.3}+\frac{3sin^3x}{1.2.4.5}+\frac{3.5sin^7x}{1.2.4.6.7}+\ldots.$$

Probl. 11. *Dato cosinu invenire arcum.*

$$u\ \text{arc}=cos\,x;\ u^1=\text{arc}\,cos(x+y).$$

Facilis per novissimum problema solutio. Prodibunt enim exponentes differentiales, forma et litteris iidem, signo tantum excepto communi, quod e positivo in negativum abit. Posito igitur $x=o$, et deinde x pro y, erit

$$u=90^0-x-\frac{x^3}{1.2.3}-\frac{3x^5}{1.2.4.5}-\frac{3.5x^3}{1.2.4.6.7}-\ldots.$$

vel, si arcus ipse vocetur x,

$$x=90^0-cos\,x-\frac{cos^3x}{2.3}-\frac{3cos^5x}{2.4.5}-\frac{3.5cos^7x}{2.4.6.7}-\ldots.$$

$=90^0-\text{arc}\,sin(cos\,x)$, ut ex ipsis principiis trigonometriæ patet.

Probl. 12. *Data tangente invenire arcum.*

Sit $u=\text{arc}\,tang\,x;\ u^1=\text{arc}\,tang\,(x+y)$, erit

$$\frac{du}{dx}=\frac{1}{1+x^2};\ \frac{d^2u}{dx^2}=-\frac{2x}{(1+x^2)^2};$$

$$\frac{d^3u}{dx^3}=\frac{-2(1+x^2)^2+8x^2(1+x^2)}{(1+x^2)^4}=\frac{6x^2-2}{(1+x^2)^3};$$

$$\frac{d^4u}{dx^4}=\frac{12x(1+x^2)^3-6x(6x^2-2)(1+x^2)^2}{(1+x^2)^6}=\frac{12x+12x^3-36x^3+12x}{(1+x^2)^4}=\frac{24x-24x^3}{(1+x^2)^4};$$

$$\frac{d^5u}{dx^5}=\frac{(24-72x^2)(1+x^2)^4-8x(24x-24x^3)(1+x^2)^3}{(1+x^2)^8}$$

$$=\frac{24+24x^2-72x^2-72x^4-192x^2+192x^4}{(1+x^2)^5}=\frac{24-240x^2+120x^4}{(1+x^2)^5};$$

qui terminorum numerus legi patefaciendæ sufficit. Facta enim theoremati Tayloriano adplicatione Mac-Lauriniana, habebimus:

$$u = \text{arc}\, tang x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots;$$

vel, si arcus adpelletur x,

$$x = tang\, x - \frac{tang^3 x}{3} + \frac{tang^5 x}{5} - \frac{tang^7 x}{7} + \ldots$$

Probl. 13. *Data cotangente invenire arcum.*

$$u = \text{arc}\, cot\, x;\ u^1 = \text{arc}\, cot\, (x + y).$$

Cum sit $d(\text{arc}\, cot\, x) = -\frac{1}{1+x^2}$, erunt exponentes differentiales, mutato signo, iidem atque in probl. 12. Hinc

$$u = 90^0 - x + \frac{x}{3} - \frac{x}{5} + \frac{x}{7} + \ldots;\ \text{sive}$$

$u = 90^0 - \text{arc}\, tang(cot\, u)$, principiis ipsis trigonometricis convenienter.

Schol. Cum in omnibus hisce quæstionibus, quoties ad computandum ventum fuerit, si sinus, cosinus, cet. ex arcu determinabitur, hic in sinum minuti secundi ($= 0{,}00000484814$) ubique ducendus sit, sin contra, sinus, cosinus, cet cum unitate, per illum sinum divisa $\left(= \frac{1}{0{,}00000484814} = 20264{,}806247\right)$, ubique multiplicandus, hi vero ipsi factores, nisi ratio diametri ad peripheriam cognita fuerit, obtineri nequeant; paucis videndum est, quomodo mutuis et recurrentibus hæc cohæreant officiis.

Probl. 14. *Dato radio, $= 1$, invenire peripheriam circuli, $= 2\pi$.*

α) E problemate 10:mo habebimus:

$$\frac{\pi}{6} = \tfrac{1}{2} + \frac{1}{3.2^4} + \frac{3}{4.5.2^6} + \frac{3.5}{4.6.7.2^8} + \frac{3.5.7}{4.6.8.9.2^{10}} + \frac{3.5.7.9}{4.6.8.10.11.2^{12}} + \ldots$$

quæ series cum non tam rapide convergat, tamen si sex tantum primorum terminorum cogantur in summam, obtinebitur $\pi = 3{,}1415$; si decem, $\pi = 3{,}1415926$.

β) Specie quidem simplicioris solutionis aderit e problemate 12

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \ldots$$

longe tamen tardiore incessu, ut, prima terminorum decade sumta, in prima nota decimali adhuc error obversetur.